Homework 6 Solutions

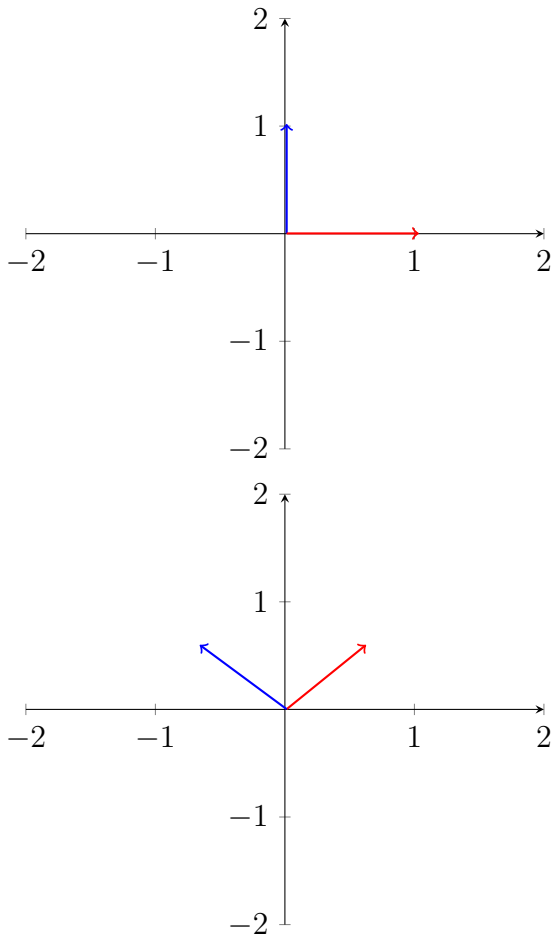
1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with associated matrix $Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Let $\theta = \frac{\pi}{4}$. Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Sketch the original vectors and images in two separate $xy-$planes. What does T do to these vectors, geometrically, in terms of θ?

   When $\theta = \frac{\pi}{4}$, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

   The input and corresponding output vectors are drawn below in red and blue, respectively.



   T rotates the vectors by an angle of $\theta = \frac{\pi}{4}$.

(b) Show that Q_θ is orthogonal using the definition of orthogonal matrices.

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the columns of Q_θ. Since $\mathbf{q}_1 \cdot \mathbf{q}_2 = \cos\theta(-\sin\theta) + \cos\theta\sin\theta = 0$, the columns of Q_θ are orthogonal. Since $||\mathbf{q}_1|| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$ and $||\mathbf{q}_2|| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1$, the columns are unit vectors. Finally, since Q_θ is a square matrix, it is orthogonal.

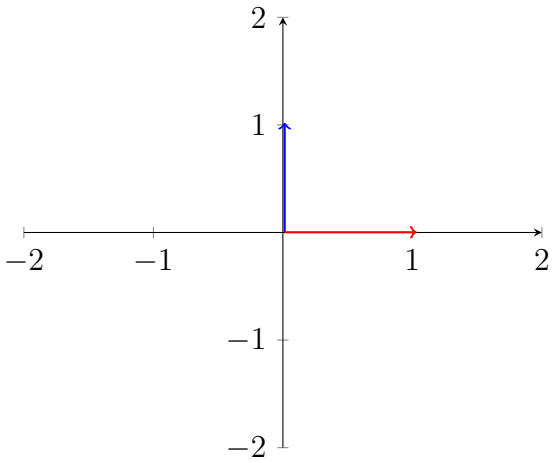
2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with associated matrix $R_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

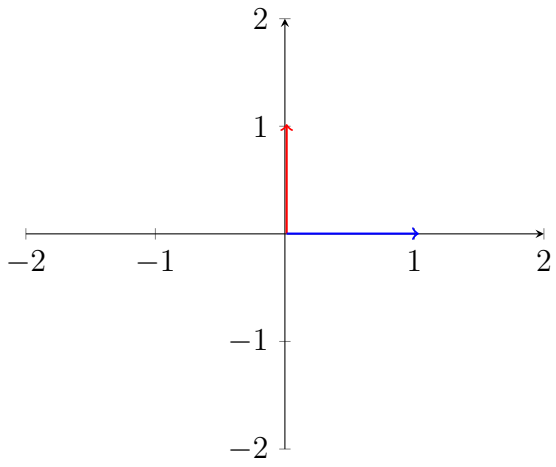
(a) Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Sketch the original vectors and images in two separate $xy-$planes. What does T do to these vectors, geometrically, in terms of θ?

When $\theta = \frac{\pi}{4}$, we have

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\frac{\pi}{2} \\ \sin\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \sin\frac{\pi}{2} \\ -\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The input and corresponding output vectors are drawn below in red and blue, respectively.





T reflects the vectors over the line $y = x$ (which has an angle of $\theta = \frac{\pi}{4}$ with the positive $x-$axis.

(b) Show that R_θ is orthogonal using the definition of orthogonal matrices.

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be the columns of Q_θ. Since $\mathbf{q}_1 \cdot \mathbf{q}_2 = \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta = 0$, the columns of Q_θ are orthogonal. Since $||\mathbf{q}_1|| = \sqrt{\cos^2(2\theta) + \sin^2(2\theta)} = \sqrt{1} = 1$ and $||\mathbf{q}_2|| = \sqrt{(\sin 2\theta)^2 + (-\cos 2\theta)^2} = \sqrt{\sin^2 2\theta + \cos^2 2\theta} = \sqrt{1} = 1$, the columns are unit vectors. Finally, since Q_θ is a square matrix, it is orthogonal.

3. The matrices Q_θ and R_θ defined in the above problems are called rotation and reflection matrices, respectively. Show that any 2×2 orthogonal matrix is either a rotation or reflection matrix.

Let Q be an orthogonal 2×2 matrix. Then since its columns are unit vectors in $\mathbb{R}^2$, we have $Q = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix}$ for some $0 \leq \alpha, \beta \leq \pi$. Since the columns must be orthogonal, we have that their dot product is zero; that is $\cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$. Since $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, we have $\cos(\alpha - \beta) = 0$. Hence $\alpha - \beta = \pm\frac{\pi}{2}$; i.e. $\beta = \alpha \pm \frac{\pi}{2}$.

First assume $\beta = \alpha - \frac{\pi}{2}$. Then $\cos\beta = \cos(\alpha - \frac{\pi}{2}) = \sin\alpha$ and $\sin\beta = \sin(\alpha - \frac{\pi}{2}) = -\cos\alpha$. Hence $Q = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = R_{\frac{\alpha}{2}}$ is a reflection matrix.

Next assume $\beta = \alpha + \frac{\pi}{2}$. Then $\cos\beta = \cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$ and $\sin\beta = \sin(\alpha + \frac{\pi}{2}) = \cos\alpha$. Thus $Q = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = Q_\alpha$ is a reflection matrix.

4. Let $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be an orthogonal basis of a subspace W of $\mathbb{R}^m$. Show that

$$\text{proj}_W \mathbf{b} = \text{proj}_{\mathbf{c}_1} \mathbf{b} + \cdots + \text{proj}_{\mathbf{c}_n} \mathbf{b}.$$

We know that $\text{proj}_W \mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b}$, where $A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$. Since $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is orthogonal, $\mathbf{c}_i \cdot \mathbf{c}_j = 0$ whenever $i \neq j$. Since the (i, j)th entry of $A^T A$ is $\mathbf{c}_i \cdot \mathbf{c}_j$, we have that $A^T A$ is the diagonal matrix

$$A^T A = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{c}_2 \cdot \mathbf{c}_2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & \mathbf{c}_n \cdot \mathbf{c}_n \end{bmatrix}$$

Note that none of the diagonal entries are 0 since $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is a basis and so none of the vectors are the zero vector. Now

$$(A^T A)^{-1} = \begin{bmatrix} 1/\mathbf{c}_1 \cdot \mathbf{c}_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/\mathbf{c}_2 \cdot \mathbf{c}_2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1/\mathbf{c}_n \cdot \mathbf{c}_n \end{bmatrix}$$

since the product of this matrix with $A^T A$ is the identity. So we have:

$$\begin{aligned}
\text{proj}_W \mathbf{b} &= A(A^T A)^{-1} A^T \mathbf{b} \\
&= \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} 1/\mathbf{c}_1 \cdot \mathbf{c}_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/\mathbf{c}_2 \cdot \mathbf{c}_2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1/\mathbf{c}_n \cdot \mathbf{c}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_n^T \end{bmatrix} \mathbf{b} \\
&= \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} 1/\mathbf{c}_1 \cdot \mathbf{c}_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/\mathbf{c}_2 \cdot \mathbf{c}_2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1/\mathbf{c}_n \cdot \mathbf{c}_n \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{b} \\ \vdots \\ \mathbf{c}_n \cdot \mathbf{b} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} \frac{\mathbf{c}_1 \cdot \mathbf{b}}{\mathbf{c}_1 \cdot \mathbf{c}_1} \\ \vdots \\ \frac{\mathbf{c}_n \cdot \mathbf{b}}{\mathbf{c}_1 \cdot \mathbf{c}_1} \end{bmatrix} \\
&= \frac{\mathbf{c}_1 \cdot \mathbf{b}}{\mathbf{c}_1 \cdot \mathbf{c}_1} \mathbf{c}_1 + \cdots + \frac{\mathbf{c}_n \cdot \mathbf{b}}{\mathbf{c}_n \cdot \mathbf{c}_n} \mathbf{c}_n \\
&= \text{proj}_{\mathbf{c}_1} \mathbf{b} + \cdots + \text{proj}_{\mathbf{c}_1} \mathbf{b}
\end{aligned}$$

5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Use Gram-Schmidt to find an orthonormal basis of W.

We first find an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

$$\mathbf{b}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{b}_2 = \mathbf{a}_2 - \text{proj}_{\mathbf{b}_1}\mathbf{a}_2 = \mathbf{b}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{b}_3 = \mathbf{a}_3 - \text{proj}_{\mathbf{b}_1}\mathbf{a}_3 - \text{proj}_{\mathbf{b}_2}\mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{-1}{3/2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

Next, we normalize these vectors to obtain an orthonormal basis:

$$\mathbf{q}_1 = \frac{1}{||\mathbf{b}_1||}\mathbf{b}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{1}{||\mathbf{b}_2||}\mathbf{b}_2 = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{q}_3 = \frac{1}{||\mathbf{b}_3||}\mathbf{b}_3 = \frac{\sqrt{3}}{2}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

(b) Use the orthonormal basis found in part (a) to find the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ onto W by using the formula in problem 4.

$$
\begin{aligned}
\text{proj}_W \mathbf{b} &= \text{proj}_{\mathbf{q}_1}\mathbf{b} + \text{proj}_{\mathbf{q}_2}\mathbf{b} + \text{proj}_{\mathbf{q}_3}\mathbf{b} \\
&= (\mathbf{q}_1 \cdot \mathbf{b})\mathbf{q}_1 + (\mathbf{q}_2 \cdot \mathbf{b})\mathbf{q}_2 + (\mathbf{q}_3 \cdot \mathbf{b})\mathbf{q}_3 \\
&= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} \\
&\quad + \left(\frac{\sqrt{3}}{2} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) \frac{\sqrt{3}}{2} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} \\
&= -\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}
\end{aligned}
$$

(c) Use part (a) to find a QR factorization of $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. Provided an exact answer (no rounding)

The columns of Q are simply the orthonormal basis vectors (in order). Hence

$$
Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} \\ -1/\sqrt{2} & 1/2\sqrt{3} & 1/2\sqrt{3} \\ 0 & -\sqrt{2/3} & 1/2\sqrt{3} \\ 0 & 0 & -\sqrt{3}/2 \end{bmatrix} \text{ and}
$$

$$
R = Q^T A = \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & \sqrt{3/2} & -\sqrt{2/3} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}
$$

(d) Use the QR factorization to find the least squares solution to $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/\sqrt{3} \end{bmatrix}$.

First note that $Q^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Hence the least squares solution is obtained

by solving $R\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, which is quick since R is upper triangular. Using back substitution yields the solution

$$\hat{\mathbf{x}} = \begin{bmatrix} -1/2\sqrt{3} \\ -1/\sqrt{3} \\ -\sqrt{3}/2 \end{bmatrix}$$

6. Let Q be an orthogonal matrix. Show that $\det Q = \pm 1$ in two ways.

   (a) Algebraically, using properties of determinants and orthogonal matrices.

   Since Q is orthogonal, $Q^T Q = I$. Now taking the determinant and using the properties $\det Q = \det Q^T$ and $\det(Q^T Q) = \det Q^T \det Q$, we have

   $$1 = \det I = \det(Q^T Q) = \det Q \det Q^T = \det Q \det Q = (\det Q)^2.$$

   Hence $\det Q = \pm 1$.

   (b) Geometrically, using intuition.

   Let T be the linear transformation associated to Q. Then $|\det Q|$ represents the volume of $T(C)$, where C is the unit cube. Since Q is orthogonal, T preserves lengths and angles; hence it must preserve volume. So, the volume of $T(C)$ is the volume of C, which is 1. Hence $|\det Q| = 1$ and so $\det Q = \pm 1$.

7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}$. Find $\det A$ using row reduction and the properties and discussed in class. Be sure to explain your steps.

Row replacement does not change the determinant and the determinant of an upper

triangular matrix is the product of the diagonal entries. Thus:

$$\det\begin{bmatrix}1&2&3&0\\2&6&6&1\\-1&0&0&3\\0&2&0&7\end{bmatrix} = \det\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&2&3&3\\0&2&0&7\end{bmatrix} \text{ (Replace R2 by R2-2R1 and R3 by R3+R1)}$$

$$= \det\begin{bmatrix}1&2&3&0\\0&2&0&1\\0&0&3&2\\0&0&0&6\end{bmatrix} \text{ (Replace R3 by R3-R2 and R4 by R4-R2)}$$

$$= (1)(2)(3)(6) = 36$$

8. Recall that the permutation matrix P_{ij} is the matrix obtained by swapping the two rows of I. On Homework 3 you showed that $P_{ij}^{-1} = P_{ij}$. Let P be a product of r permutation matrices (P is also called a permutation matrix).

   (a) Use determinant axioms/properties discussed in class to show that $\det P = (-1)^r$.

   First note that by axioms 1 and 2, $\det P = -\det I = -1$. Thus

   $$\det P = \det(P_1 \cdots P_r) = \det P_1 \cdots \det P_r = (-1)^r.$$

   (b) Show that $P^{-1} = P^T$

   We first show that $P_{ij}^T = P_{ij}^{-1}$. To do this, we'll show that $P_{ij}P_{ij}^T = I$. Without loss of generality, we may assume that $j > i$. Notice that $I = \begin{bmatrix}\mathbf{e}_1^T\\ \vdots \\ \mathbf{e}_i^T \\ \vdots \\ \mathbf{e}_j^T \\ \vdots \\ \mathbf{e}_n^T\end{bmatrix}$; hence $P_{ij} = \begin{bmatrix}\mathbf{e}_1^T\\ \vdots \\ \mathbf{e}_j^T \\ \vdots \\ \mathbf{e}_i^T \\ \vdots \\ \mathbf{e}_n^T\end{bmatrix}$.

   Thus the product is

$$P_{ij}P_{ij}^T = \begin{bmatrix}\mathbf{e}_1^T\\ \vdots \\ \mathbf{e}_j^T \\ \vdots \\ \mathbf{e}_i^T \\ \vdots \\ \mathbf{e}_n^T\end{bmatrix}\begin{bmatrix}\mathbf{e}_1 & \cdots & \mathbf{e}_j & \cdots & \mathbf{e}_i^T & \cdots \mathbf{e}_n\end{bmatrix} = \begin{bmatrix}\mathbf{e}_1\cdot\mathbf{e}_1 & \mathbf{e}_1\cdot\mathbf{e}_2 & \cdots & \mathbf{e}_1\cdot\mathbf{e}_n\\ \mathbf{e}_2\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_2 & \cdots & \mathbf{e}_2\cdot\mathbf{e}_n\\ \vdots & & & \vdots \\ \mathbf{e}_n\cdot\mathbf{e}_1 & & \cdots & \mathbf{e}_n\cdot\mathbf{e}_n\end{bmatrix} = I,$$

where the last equality follows from the fact that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for all $i \neq j$ and $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ for all i.

Now let $P = P_1 \cdots P_r$, where each P_i is a permutation matrix that swaps two rows. Then since they are all invertible,

$$P^{-1} = (P_1 \cdots P_r)^{-1} = P_r^{-1} \cdots P_1^{-1} = P_r^T \cdots P_1^T = (P_1 \cdots P_r)^T = P^T.$$

(c) Use the above parts to show that $\det P = \det P^T$.
[This is used to prove the general property $\det A = \det A^T$ (see lecture notes); so you cannot use the general property to complete this part].

By part (a), $\det P = (-1)^r$.
By part (b), $P^{-1} = P^T$ and so $P^T P = I$. It follows that

$$(\det P)(\det P^T) = \det PP^T = \det I = 1.$$

Hence $\det P^T = \frac{1}{\det P} = \frac{1}{(-1)^r} = (-1)^r = \det P$.